NATIONAL BUREAU OF STANDARDS REPORT

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SOME RECENT THEORETICAL WORK ON ACCESSIBILITY IN TRANSPORTATION SYSTEMS

W.A. Horn

U.S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

NATIONAL BUREAU OF STANDARDS

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SOME RECENT THEORETICAL WORK ON ACCESSIBILITY IN TRANSPORTATION SYSTEMS

W.A. Horn

Applied Math Division

Technical Report

to

Northeast Corridor Transportation Project

Department of Transportation

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"SOME RECENT THEORETICAL WORK ON ACCESSIBILITY IN TRANSPORTATION SYSTEMS"

Inner Title Page: DATE, December 1970

Abstract, line 2: reviewed $\rightarrow$ summarized

Page 15, line 3: constraints is $\rightarrow$ constraints, is

Page 16: INSERT between paragraphs 2 and 3 the following paragraph:

> The next step is to determine a set of subregions of the given regions (each subregion being the intersection of some subcollection of the collection of regions), such that the constraints will be satisfied if each such subregion contains an access point.

Page 16, line 7^-: regions $\rightarrow$ subregions

line 6^-: region $\rightarrow$ subregion

line 2^-: example. $\rightarrow$ example,

Page 17, line 6: negligible

Page 25, line 4^-: $t^o_q \rightarrow t_q{}^o$

line 3^-: $F(q^o) - F(q^*) \rightarrow F(q^o) - F(q^*)$

FOOTNOTE: 1969 $\rightarrow$ 1959

Page 37, No. 20, line 2: *Amer. J. Math.*

No. 21, line 5: *Operations Research*

SOME RECENT THEORETICAL WORK ON ACCESSIBILITY IN TRANSPORTATION SYSTEMS

W.A. Horn

ABSTRACT

Four NECTP-sponsored papers, which have already been published in some form, are reviewed in this paper. A fifth, unpublished, work is also presented in its entirety. A rough theoretical framework is constructed to define accessibility in transportation systems, and unsolved problems of interest are enumerated. A short bibliography of relevant work is also included.

Key words: Transportation systems, transportation networks, weighted graphs, accessibility, minimum cover problems, location theory

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1. INTRODUCTION

In the theoretical study of transportation systems, one naturally encounters the concept of <u>accessibility</u>. The following is an attempt to summarize the various pieces of research relating to accessibility performed in support of the Northeast Corridor Transportation Project (NECTP) during the years 1967-69. Part of the work discussed was done at the National Bureau of Standards (NBS), and the rest was monitored or reviewed by the staff at NBS.

This paper will begin with some general comments on the types of mathematical problems presented by the study of accessibility, and will then turn to a description of the specific work done. Suggestions for further research appear next, followed by a brief bibliography.

First we define accessibility in a way which, while somewhat general, presages certain of the mathematical problems to be encountered. The accessibility problem, from a mathematical point of view, concerns <u>the geographic placement of a transportation system, or part of a transportation system, so as to achieve the most convenience for the users of this system at the least cost</u>.

The system (or part of a system) in question will be the "track" or "road-bed" on which vehicles travel, in the case of land vehicles, or the points of entrance to and exit from the system in the case of all types of transportation. The goal of maximizing user convenience while minimizing cost will give rise to three types of mathematical problems:

a) <u>Fixed-Budget Problem</u>. Maximize user convenience subject to fixed cost constraints which may not be exceeded.

b) Convenience-Constrained Problem. Devise a least-cost system which will achieve at least some minimum amount of convenience for each user of the system.

c) Tradeoff Problem. Find a system which minimizes some weighted combination of cost and user inconvenience.

Each of these problems may involve additional, complicating constraints. For example problem a) may have the additional constraint that some minimum level of convenience must be attained for each user, although the overall objective is to maximize a total convenience measure for all users. Sometimes a problem posed in these terms has no solution (e.g., where the prescribed budget limit is too low to permit meeting the minimum-convenience conditions).

More specifically, the following problems come to mind as having significance in the area of accessibility.

1) Determine the location of a limited number of terminals (or access points) for a system in such a way that total time spent in getting to the nearest terminal is minimized.

2) Same as 1), but minimize total time in getting to terminal plus time to destination.

3) Determine the number and placement of terminals so as to require no more than a given amount of time loss to each user in getting to the system.

4) Determine the locations of a given number of terminals so as to minimize the greatest access time loss suffered by any user.

5) Any of problems 1), 2), or 4) with a tradeoff for cost of system.

6) Given a set of terminal locations, determine the configuration of a system which minimizes construction cost.

7) Same as 6), but add in tradeoff factor for inconvenience of using a given system.

8) Find a set of terminal locations and a configuration of the system which minimizes some weighted sum of cost and inconvenience.

9) Same as 3), but also determine the rest of the system, given the placement of terminals, so as to minimize some cost/inconvenience function.

Four of the five papers discussed in this report (and supported by the NECTP) fall under categories 1), 5), 7), and 9), while a fifth is too theoretical to be readily put in a specific category. Four of the above papers have been published already in some form and are summarized in section 2. Another paper, unpublished, is presented in full in section 3.

Section 4 contains recommendations for possible future work, while section 5 lists the bibliography of reference cited, and also some other literature of possible interest to one working in the field of accessibility. For example [18], [19], [21] and [22] pertain to categories 1) and 4), [17] to category 2), and [20] and [23] to category 3).

2. A SUMMARY OF NECTP-SPONSORED PAPERS

2.1. OPTIMAL NETWORKS JOINING n POINTS IN THE PLANE, by W.A. Horn

The report version of this paper was sent to D.O.T., and a condensed form has been published [10] in the Proceedings of the Fourth International Symposium on the Theory of Traffic Flow, held in the summer of 1968 in Karlsruhe, Germany. An expanded version will be published in the NBS Journal of Research at a later date.

The paper deals with the following problem. Given a (finite) set of n initial points in the plane, find a network N connecting these points which minimizes the function

$$f(N) = \lambda \ell(N) + \sum_{i,j} \lambda_{ij} \ell(P_{ij}) ,$$

where $\ell(N)$ is the total length of all arcs of N, $\ell(P_{ij})$ is the length of some shortest path P_{ij} in N connecting initial points i and j, and λ and λ_{ij} are, respectively, positive and non-negative parameters. With reference to the discussion of section 1, this problem is of type 7), where λ represents construction cost per unit length of arcs in the network and λ_{ij} represents the "inconvenience" per unit length for travelers using the network from i to j; thus λ_{ij} is generally taken as proportional to the number of travelers between i and j.

This paper is more theoretical than directly-applicable, although the problem is solved completely for $n=3$ and a solution algorithm is suggested for larger n, which might be feasible on a computer for values of n not much larger than 3. However, the main contribution of the paper is to determine characteristics of such networks for general n. The following are the chief results obtained.

1. An optimal network consists of a connected set of straight line segments each of which is contained in the convex hull of the original set of points (initial nodes) and must be a part of some minimal (shortest) path P_{ij} under any possible assignment of minimal paths within the optimal network.

2. The n initial nodes of an optimal network are of order at most $n-1$, while all other (auxiliary) nodes are of order at most n. Furthermore, every arc incident at an initial node must carry at least one minimal path from that node to some other initial node in any possible assignment of minimal paths within the network.

3. The angles made by arcs at any auxiliary node are dependent on the weights (sum of λ and of all λ_{ij} for which minimal path P_{ij} lies on the arc) of the arcs. Thus if the k-th arc at a given auxiliary node has weight w_k and makes an angle σ_k with some fixed ray emanating from the auxiliary node, then we have

$$\Sigma_k w_k \cos \sigma_k = 0 .$$

4. There exists an optimal network for any set of n initial nodes, and any such optimal network contains at most

$$\tfrac{1}{4}n(n-1)^2(n-2)$$

auxiliary nodes.

5. The only possible optimal networks on 3 initial nodes are of the topological type shown in Figure 1, where configuration (e) need not be considered since its existence implies the existence also of one of the other types as optimal.

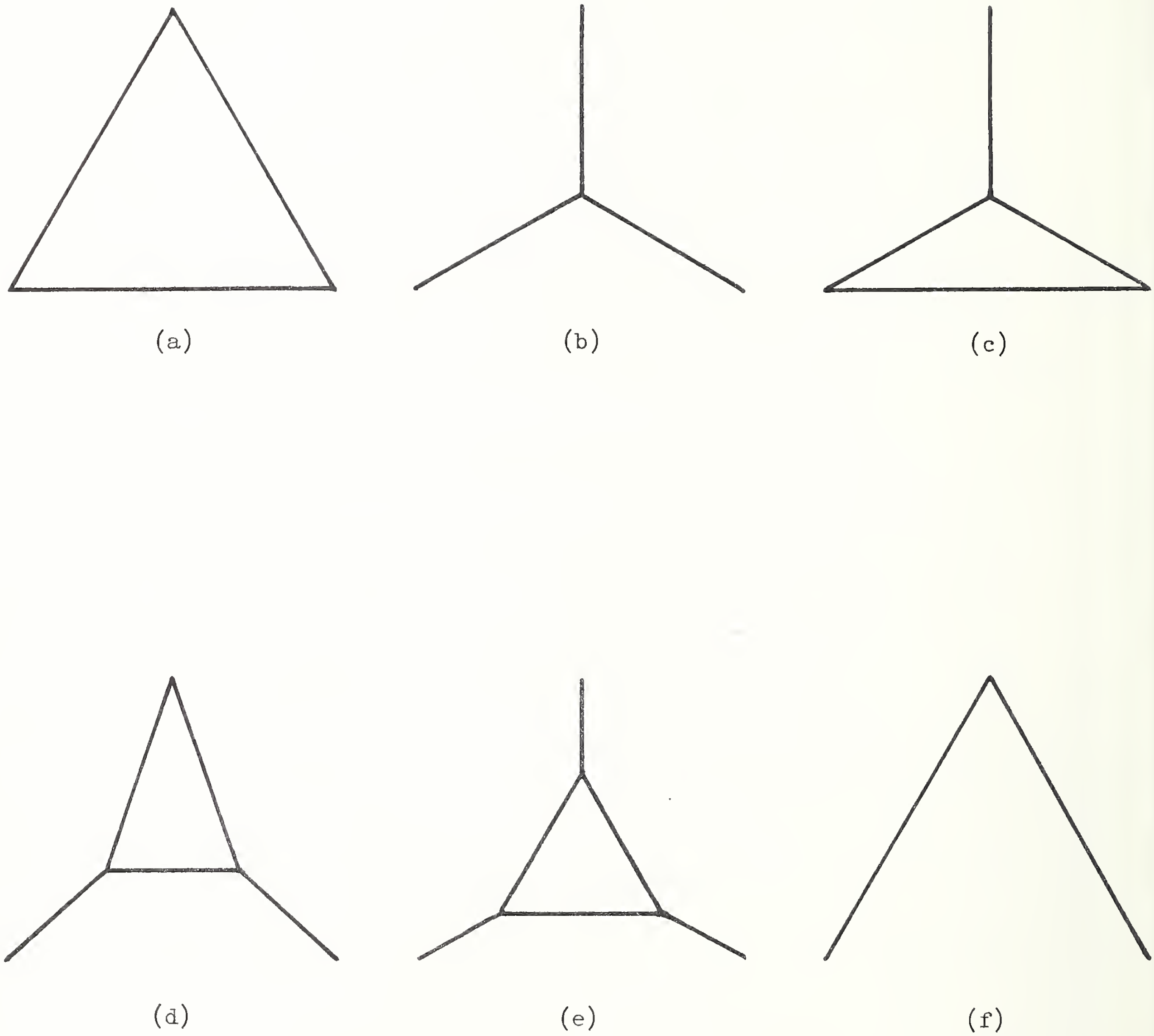

Figure 1. Six possible optimal networks for n=3

6. If $\lambda/(\lambda+\Sigma\lambda_{ij}) > \sqrt{2}-1$, then any optimal network contains no nodes of order greater than 3 and no triangles or quadrilaterals; therefore every auxiliary node is of order 3 .

Other results concerning specific configurations are given to aid in determining the optimal networks for small n .

2.2. INTERSTATION SPACINGS FOR LINE-HAUL PASSENGER TRANSPORTATION, by Vukan R. Vuchic

The following is a brief account of the most valuable contributions to be found in Vuchic's paper [14]. Some liberty has been taken in generalizing certain of the ideas and in modifying the solution procedure to provide what is hoped to be a more useful approach. The paper is of type 5).

Assumptions for the model are these: A single transportation line is planned along a fixed route of given length, to serve customers along the line and in its surrounding area who wish to travel to a central business district (CBD) at the end of the line. The form of transportation used is considered to be rail, with a limited number of access points (stations). It is assumed that all passengers ride to the end of the line (CBD) after boarding.

The population desiring to use this system is fixed, and the density function of its geographic distribution is known. In addition it is assumed that there is a "nearest" or "most desired" point of access to the system for each user, such that this user will enter the system at this point, if occupied by a station, and otherwise at one of the two station locations which are nearest this point in either direction along the line.

[This last condition was achieved in [14] in the following way. It was assumed that the line itself runs along the x-axis of a coordinate system, while access travel must be performed at constant speed along roads parallel to the x- or y-axis. With total travel time as the criterion for selection of the access point for an individual traveler, it is clear that the traveler must choose one of the two stations with x-coordinate closest to his own. For he first travels in a y-direction to the line and then must head toward or away from the CBD. If he travels away, it is clear that the first station he comes to is best for access, while if he travels toward the CBD the first station encountered must also be the best in that direction, since otherwise one could go from this station to some later station by means of the access road in shorter time than by taking the train, implying the uselessness, and hence lack of need for, this station.]

A further assumption for the model is that the average speed of the train is not necessarily the same between different pairs of stations, but is a monotonic increasing function of the distance of travel between stations, which approaches the cruising speed of the train asymptotically. This is due to acceleration-deceleration effects at the stations. The assumption gives rise to a minimum distance between stations below which train speed is less than access-road speed.

The objective function, to be minimized, for the above model is the total cost for the system, consisting of a construction cost for road bed (variable with the positioning of the first station), a cost for stations, and a cost for total time spent by all users of the system. This objective function is taken from Chapter IV of [14], although the method of solution described below will be an adaptation of the methods of Chapter III.

Measuring distance along the proposed line, from an origin at the end of the line farthest from the CBD, let

x_1 = distance of the first station from the origin,

k_ℓ = cost per unit length of line,

k_s = cost per station,

k_p = cost per unit of passenger time spent in travel,

n = number of stations,

T_p = total passenger time.

Then an appropriate objective function is

$$C = -k_\ell x_1 + k_s n + k_p T_p ,$$

where the first term arises because no roadbed need be put between 0 and x_1 on the line.

A dynamic program which is an adaptation of the recursion formula of Chapter III of [14] will be used to solve this problem. We assume that there are a number of locations $L_1, L_2, \ldots, L_m$ along the line, with respective distances $d_1, d_2, \ldots, d_m$ from the origin, where it is possible to locate a station. Let

> R_i = region in which each user has a nearest point on the line with coordinate lying between 0 and d_i, inclusive.

That is, if $d(p)$ is the distance from the origin to the desired access point for a passenger at p,

$$R_i = \{p : 0 \leq d(p) \leq d_i\} .$$

Let

> $C_j(i)$ = minimum cost for travel time and for construction of stations and line, for the portion of the system from 0 to d_i and for all passengers in R_i, assuming station j is located at L_i and all prior stations (1 through j-1) are located optimally along the strip $[0, d_{i-1}]$. (The "initial condition" is $C_o(i) = 0$.)

That is, if x_1 is the location of station 1 and if T_i^j is the total passenger time spent in travel up to the point d_i by passengers in R_i (all of whom have boarded the system by d_i), assuming station j is located at L_i and stations 1 through j-1 are located optimally, then

$$C_j(i) = -k_\ell x_1 + k_s j + k_p T_i^j \ .$$

Note that $C_j(i)$ does not in any way depend on the location of later stations, and hence is well defined as given.

Now let P_i be the number of people living in R_i and let $t(i,k)$ be the time required for a train to go from a station at L_i to a station at L_k. Let S_{ik} be the total waiting time for all passengers in $R_k - R_i$ to arrive at point L_k, assuming stations are at L_i and L_k with none in between. Then we have the recursion relation

$$C_{j+1}(k) = \min_{i \leq k-1} \{C_j(i) + k_s + k_p[P_i t(i,k) + S_{i,k}]\} \ .$$

It should be noted at this point that the quantity $S_{i,k}$ will in general involve some assumption as to how the residents in $R_k - R_i$ would choose between the stations at L_i and L_k for access to the system. It will be assumed that this choice depends only on the values of i and k, as is the case in Chapter III of [14], where minimum time to arrive at a station is the sole criterion for each user.

The total cost for n stations, the last of which is located at the CBD (also designated L_m), is then

$$C = C_n(m) \ .$$

Of course, if n is not fixed then we look for an n which minimizes $C_n(m)$.

Some of the calculations in the recursion step may be omitted by stipulating, for example, a minimum distance between successive stations.

It seems that the above method should prove feasible for solving problems of reasonable size on the computer.

Finally, it should be noted that Vuchic found, using a uniformly distributed population and objective function based on waiting time only, that station spacing should increase as one nears the CBD. This corresponds to what seems logical, because of the increasing number of riders as the train progresses.

2.3 OPTIMUM ALLOCATION OF TRANSPORTATION TERMINALS IN URBAN AREAS

by Barton E. Cramer

The following problem is considered in this paper [4]. Given a city which is circular in shape, and given the number of terminals for a transportation network which is to serve the city, place these terminals so as to minimize total access time between users and nearest terminals. This is of type 1) of section 1.

The assumption is made that, in the circular city, population density is a function only of distance from the center, as is also speed of travel, which increases monotonically with distance from the center, approaching asymptotically some fixed value. Furthermore, the only configurations of terminals allowed are the following: a possible terminal at the center plus one or two concentric rings of equally spaced terminals. In the event that two rings of terminals are considered, the outer ring is assumed to contain the same number of terminals as the inner ring, spaced angularly between them. That is, if the terminals of the inner ring are placed at angles $0, \theta, 2\theta, \ldots, (n-1)\theta$, then the terminals of the outer ring have angular coordinates $\theta/2, 3\theta/2, 5\theta/2, \ldots, (2n-1)\theta/2$.

Computer results are derived in this paper, but their possible significance is nullified by errors in the underlying algebra. Thus we must evaluate the paper on its approach to the problem rather than its results, perhaps a more appropriate criterion for this type of paper (a master's thesis) anyway.

Probably the greatest merit of the approach taken lies in the fact that, because of the restricted nature of possible solutions, it should not be too difficult to obtain a computer optimization

of the problem. If the assumptions about population density and travel speed are reasonably accurate, it might happen that the answer obtained in this way would compare favorably with that obtained by more elaborate methods. On the other hand, there may be other models of a similar order of complexity which yield more practical solutions, such as a model which presents a limited number of choices for terminal sites and chooses an optimum subset of these for the actual locations. Such a model could also condider the finer points of locating a set of terminals in a real-life siutation (such as real estate cost, neighborhood opposition or approval, etc.), while retaining the advantages of a restricted set from which to select terminals.

2.4 MINIMUM-LENGTH COVERING BY INTERSECTING INTERVALS, by W.A. Horn

This paper was published [9] in the NBS Journal of Research and so it will be merely summarized here. Because of its theoretical nature it cannot easily be placed in one of the nice categories of section 1, although (as discussed in [9]) it arose from a problem of type 6), namely that of finding a shortest-length network of "vertical" and "horizontal" lines connecting a set of points consisting of several well-separated "clusters", each on a vertical line. The problem considered is the following. Given a sequence $\{I_i\}_1^n$ of intervals on the real axis, find a sequence $\{J_i\}_1^n$ of closed intervals which minimizes the sum of lengths

$$S = \Sigma_1^n |J_i|$$

subject to the constraints

$$I_i \subset J_i \qquad (i=1,2,\ldots,n) ,$$

$$J_i \cap J_{i+1} \neq \emptyset \qquad (i=1,2,\ldots,n-1) .$$

An algorithm which provides an optimal solution is the following. Set $J_1 = I_1$. Given $\{J_j\}_1^i$, J_{i+1} is found as follows:

(a) If $J_i \cap I_{i+1} \neq \emptyset$, set $J_{i+1} = I_{i+1}$.

(b) If $\max J_i < \min I_{i+1}$, set

$$J_{i+1} = [\max J_i, \max I_{i+1}] .$$

(c) If $\max I_{i+1} < \min J_i$, set

$$J_{i+1} = [\min I_{i+1}, \min J_i] .$$

The paper also notes that the problem

$$\min \qquad S = \sum_1^n \alpha_i |J_i| \,, \qquad (\alpha_i > 0)$$

subject to the previous constraints is easily solvable by a linear program whose variables are the endpoints of the J_i . Thus if $J_i = [A_i, B_i]$, and $I_i = [a_i, b_i]$, the problem becomes

$$\min \qquad S = \sum_1^n \alpha_i (B_i - A_i)$$

subject to

$$A_i \leq a_i \,, \; B_i \geq b_i \quad (i=1,2,\ldots,n) \,,$$

$$A_i \leq B_{i+1} \,, \; B_i \geq A_{i+1} \quad (i=1,2,\ldots,n-1) \,.$$

3. LAYOUT OF A TRANSPORTATION NETWORK WITH DOMINANT TERMINAL COSTS by W.A. Horn

3.1 INTRODUCTION

The problem of planning both the layout of a transportation network, and the locations of its access points, is complicated by the need to look at both factors simultaneously. This paper considers the problem under a basic assumption which permits a simplifying sequential treatment. The assumption is that the "per access point" portion of total variable system cost is so dominant, that the first consideration is to minimize the number of access points.

This minimization must, of course, be subject to constraints which ensure adequate access to the network. Such constraints will be expressed, below, by specifying a list of (overlapping) regions, each of which is required to have at least one access point located within it. These regions might be defined in terms of lying "close enough" to some population concentration to be serviced by the system, but are open to other interpretations.

Once a set of regions has been determined, the remaining secondary problem is to locate specific points (one per region) and connect these points by a transportation network so as to minimize the total cost of such a network. This cost might be idealized as proportional to the length of the network or, for example. might also include a component based on the expected use of the network between points.

The topological configuration of such a network assumed for this paper is that of a single path consisting of straight line segment between access points, starting at one such point and passing through all other access points to end at some final access point. In using this topology, we are, in effect, assuming that the cost of the network itself is neigligible compared to transportation costs, so that it is always economical to install a direct path between two adjoining access points in a given ordering.

On the other hand, one might make the assumption that only the length of the network is important, thereby giving rise to a Steiner-minimal-tree problem. (This is the problem of 2.1, with all $\lambda_{ij} = 0$.) A more complex alternative gives weights to both the length of the network and the lengths of individual paths between points. This problem is much more complex, and is related to, but not the same as, the general problem of 2.1.

The mathematical procedure described leads naturally to a computer program for solving both the primary and secondary problems in a reasonable computation time. The particular methods outlined are in fact of an essentially brute-force type, but this should not be troublesome in the context of likely applications, involving a relatively small number of constraint regions (perhaps about 20) and thus a small minimized number of access points (hopefully, only 5 to 10). Trial computations should be helpful in indicating the problem sizes for which the proposed method becomes ineffectual and more powerful procedures must be sought.

3.2. THE PRIMARY PROBLEM: MINIMIZING THE NUMBER OF ACCESS POINTS

The mathematical formulation of this problem is as follows. Given a collection $\{R_i\}_1^n$ of n regions in the plane, find sets $P = \{p(j)\}$ of points in the plane, containing as few points as possible, such that each R_i contains at least one $p(j)$. The R_i, which need not be disjoint, correspond to the regions to be served; the $p(j)$ represent the access points.

The problem can be translated into an equivalent purely combinatorial form. For this purpose, let N_n denote the set $\{1,2,\ldots,n\}$ of integers, and let Σ denote the family of all subsets of N_n of the form

$$S_x = \{i \in N_n : x \in R_i\} \qquad (x \in \cup_1^n R_i) .$$

Note that Σ is necessarily finite, and is a <u>cover</u> of N_n in the sense that $\cup\Sigma = N_n$. The problem of finding a minimum set P of access points will now be shown equivalent to the <u>minimum cover problem</u> of finding among those subfamilies Σ_0 of Σ which, like Σ, are covers of N_n, a subfamily with as few members as possible.

First, consider any $P = \{p(j)\}_1^m$ such that each R_i contains at least one $p(j)$. This last condition is equivalent to the assertion that the subfamily $\{S_{p(j)}\}_1^m$ of Σ is a cover of N_n, and hence a candidate to solve the minimum cover problem.

On the other hand, any subfamily Σ_0 of Σ can be written in the form

$$\Sigma_0 = \{S_{p(j)}\}_1^m .$$

for some set of points $\{p(j)\}_1^m$ in $\cup_1^n R_i$,

and requiring Σ_o to be a cover of N_n is equivalent to requiring that each R_i contain at least one $p(j)$.

Therefore, the set of all minimum covers $\Sigma_o \subset \Sigma$ gives rise to all minimum sets $P = \{p(j)\}$ of access points, and conversely.

The minimum cover problem is a well known and difficult topic in combinatorial optimization. Sophisticated solution methods have been developed for larger problems ([6], [13]). For present purposes, however, it seems simplest to recall (proof to follow) that minimum cover problems are a special case of integer linear programming problems, and to rely upon the methods [1] developed for the latter broader class of problems. Whatever method is employed, the following two elementary devices for preliminary simplification may prove helpful. To describe them, we suppose an enumeration $\{S^k\}_1^K$ of Σ is at hand:

(a) If two members of Σ , S^k and S^j , are such that $S^k \subset S^j$, then S^k can be discarded without prejudice to the goal of finding a solution to the minimum cover problem. (This may however affect the secondary problem, as will be discussed later.)

(b) There may be "clusters" of regions R_i which are disjoint from all other regions; in this case the problem admits a corresponding decomposition. Specifically, suppose there is a partition

$$P = \{T_u\}_1^m \text{ of the set of integers } N_n ,$$

where

$$i \epsilon T_u , \quad j \epsilon T_v , \quad u \neq v$$

implies that R_i and R_j are disjoint in the plane. Then it is not hard to show that any minimum cover $\Sigma_o = \{S^k\}$ can be partitioned into a set of subsets $\{\Sigma_u\}_1^m$ of Σ_o in such a way that Σ_u is a minimum cover for T_u . Conversely any solutions to the minimum cover problem for the subsets T_u can be amalgamated to yield a solution to the original problem.

The reformulation of the minimum-cover problem involving N_n and $\Sigma = \{S^k\}_1^K$ as an integer linear programming problem goes as follows. Let $\{x_k\}_1^K$ be a set of 0-1 variables corresponding to the sets S^k in Σ, with the interpretation that $x_k = 1$ if S^k is part of a solution (minimum cover) and $x_k = 0$ otherwise. Let (a_{ik}) be an $n \times K$ matrix defined by $a_{ik} = 1$ if $i \in S^k$ and $a_{ik} = 0$ otherwise. Then the requirement of covering N_n is expressed by the constraints

$$\Sigma_{k=1}^{K} a_{ik} x_k \geq 1 \qquad (i=1,2,\ldots,n), \tag{1}$$

while the desire for a minimum cover is expressed by the objective

$$\text{minimize} \quad \Sigma_1^K x_k \, . \tag{2}$$

This integer linear program can be solved by whatever method is implemented in a conveniently available computer code. Such a method might be of the "cutting-plane" type, an extension to procedures for solving continuous-variable linear programs, or might be of the "branch and bound" variety, here involving implicit enumeration of all covers contained in Σ together with tests which eliminate more and more alternatives from full examination as the search for a minimum proceeds. It should be noted, however, that all of these methods aim mainly at finding one optimal solution, whereas for use with the secondary problem we should prefer to find all such optima or at least a reasonable sample of them. Thus some addition to or modification of a standard method seems to be called for.

If the problem is small enough (as we assume here), then the following brute-force recursive procedure is useful for finding all minimum covers. Consider a row of the matrix (a_{ik}) which has the least number of non-zero a_{ik}. If this number is 1, then clearly that $x_k = 1$ for which $a_{ik} = 1$, since otherwise constraint (1) is not

satisfied for row i . If the non-zero a_{ik} total more than 1, choose some $a_{ik} = 1$ and set the corresponding $x_k = 1$. Now eliminate column k and all rows j such that $a_{jk} = 1$.

Solve this subproblem by a repetition of the above methods and combine a solution of the subproblem with $x_k = 1$. This gives a minimum cover over the subset of all covers including S^k . Thus the final step in the procedure is to choose all k such that $a_{ik} = 1$ and <u>such that</u> a cover containing S^k which is minimum for all such covers is minimum over the class of all covers (including those not containing S^k). This procedure produces all minimum covers.

Before ending this discussion of the primary problem (finding a minimum set of access points), two more topics require treatment. <u>First</u>, we have not as yet described a good practical way to find the family Σ from the given set $\{R_i\}_1^n$ of planar regions. Two approaches seem plausible here. In the first, a grid of points x is superimposed on the area $\bigcup_1^n R_i$ under consideration. For each point x in turn, the set S_x is determined, and is added to the list of Σ-members if it duplicates no current member of that list. (If it is a proper subset of some current member, it might also be discarded, in view of the simplification step (a) noted earlier.) For an alternative approach, define a subset S of N_n to be <u>allowable</u> if

$$\cap \{R_i : i \epsilon S\} \neq \phi ,$$

and to be <u>maximal</u> if it is allowable but adding any element of $N_n - S$ to it destroys allowability. Then one might seek to generate all maximal subsets of N_n ; these will not constitute all of Σ , but the missing members would be eliminable by simplification step (a) anyhow.

Second, some justification should be given for the repeated suggestion that the primary problem is only of moderate size, hence tractable to solution methods of a relatively brute-force kind. There are two parameters which roughly indicate the computational complexity of a minimum-cover problem. One is the size of the set to be covered; here that set is N_n, of size n, and we have already indicated $n = 20$ as an anticipated typical value. The other is the size of the family Σ from which the minimum cover is to be chosen. To estimate this parameter, assume $n = 20$.

The maximum number of sets intersecting any given set will vary, of course, with the size and shape of the R_i. However, the number 5 as an average does not seem too far off for reasonable shapes and sizes. Thus for getting a rather rough estimate of the number of S_k's, we assume that each R_i intersects exactly 5 other R_i's.

In this case, there will be ${}_5C_q$ S_k's of cardinality $q + 1$ containing any given $i \in N_n$. If we count the S_k's by considering each $i \in N_n$ and its associated S_k's, then each S_k of cardinality $q + 1$ will be counted $q + 1$ times. Since there are (presumably) 20 such i, our total number of S_k is

$$20 \sum_{q=0}^{5} ({}_5C_q)/(q+1) = 210 .$$

This gives a rough idea of the number of S_k for a reasonable size problem and hence indicates the difficulty with an integer program solution.

As a final remark, we note that the problem admits an easy solution in one dimension. This is given in the Appendix, 3.4. Considering a given problem to be one-dimensional might be a realistic simplification, for example, if one is dealing with a corridor-shaped area in which the R_i are as wide, or almost as wide, as the area itself.

3.3 THE SECONDARY PROBLEM: NETWORK LAYOUT

The approach described in the preceding section provides us with solutions $\Sigma_0 = \{S_j\}_1^m$ of the minimum cover problem. To each $S_j \in \Sigma_0$ there corresponds the region

$$C_j = \bigcap \{R_i : i \in S_j\}$$

in the plane. Any choice of points $p(j) \in C_j$ leads to a minimal set $P = \{p(j)\}_1^m$ which meets every R_i .

Thus for the secondary problem, we assume given a collection of disjoint sets $\{C_j\}_1^m$. For the methods of this section, each C_j will be assumed to be a convex polygon; this will automatically be true if all R_i's were such polygons, but otherwise C_j will be approximated by a polygon contained in, and having vertices on the boundary of, the original C_j . We are to find an ordering of the C_j's , and points $p(j) \in C_j$, so as to minimize some function dependent both on the ordering and on the sum of distances between $p(j)$'s adjacent in the ordering. This function is intended to express costs for connecting the set of points $p(j)$ in the given order. If an ordering is represented by a permutation π of $\{1,2,\ldots,m\}$, where $C_{\pi(j)}$ is to be the j-th member of the ordering then the function to be minimized takes the form

$$f(\pi) = g(\pi) + \Sigma_1^{m-1} h_{\pi(j)\pi(j+1)}(|p(\pi(j+1)) - p(\pi(j))|) ,$$

where $|p-p'|$ denotes the distance between points p and p' .

Our approach to this problem, too, is essentially a brute force technique. It consists of optimizing the placements $p(j) \in C_j$ separately for each permutation π , and then comparing the resulting minimum values to find the best π . Since the number

of permutations π is $(m!)/2$ (the factor 1/2 being present because reversing the ordering yields the same solution), this method depends for its practicality on the minimized number (m) of access points being fairly small. On the other hand, for the roughly linear configuration of a Corridor-type area, relatively few of the possible permutations π will represent plausible candidates corresponding to reasonable network configurations.

For a given permutation π, let

$$h_j = h_{\pi(j)\pi(j+1)} , \quad q_j = p(\pi(j)) , \quad A_j = C_{\pi(j)} .$$

Then the problem to be solved is that of choosing

$$q = (q_1,\ldots,q_m)$$

to minimize

$$F(q) = \Sigma_1^{m-1} h_j(|q_{j+1}-q_j|)$$

subject to $q_j \epsilon A_j$ for $j = 1,2,\ldots,m$.

Since $h_j(x)$ represents the cost associated with a link of length x between polygons A_j and A_{j+1}, it is reasonable to assume that each function h_j is monotone increasing. We shall also assume it convex and continuously differentiable. This implies that $F(q)$ is also convex, since if q' and q'' are any two values of q, if $i \leq j \leq m$, and if a' and a'' are non-negative numbers summing to unity, then

$$\begin{aligned}
& h_j(|a'q'_{j+1}+a''q''_{j+1}) - (a'q'_j+a''q''_j)|) \\
= \quad & h_j(|a'(q'_{j+1}-q'_j) + a''(q''_{j+1}-q''_j)|) \\
\leq \quad & h_j(a'|q'_{j+1}-q'_j|+a''|q''_{j+1}-q''_j|) \\
\leq \quad & a'h_j(|q'_{j+1}-q'_j|) + a''h_j(|q''_{j+1}-q''_j|) .
\end{aligned}$$

The convexity of $F(q)$ can be used to justify the following solution method by <u>cyclic minimization</u>. The variables q_j are considered in cyclic order. At the t-th stage, one has a current estimate $q^{(t)}$ of the solution. If q_j is the distinguished variable at this stage, then hold all other variables $\{q_J : J \neq j\}$ fixed at their values $q_J^{(t)}$ in $q^{(t)}$, so that $F(q)$ becomes a function of q_j alone; find $\bar{q}_j \epsilon A_j$ to minimize this function, form $q^{(t+1)}$ from $q^{(t)}$ by changing the j-th coordinate to $\bar{q}_j$, and proceed to the next stage.

That this process converges to some point q^* can be proved by an appropriate modification of a known method+ for the case in which variables q_j are one-dimensional rather than two-dimensional. The basic idea of the argument is that the sequence $F(q^{(t)})$ is non-increasing and bounded from below, and therefore has a limit. By the compactness of the region in which q varies, the sequence $q^{(t)}$ has at least one limit point q^* for which $F(q^*)$ is the limit of $F(q^{(t)})$. It is only necessary to show that q^* gives the minimum for $F(q)$, as will now be done.

Suppose, to the contrary, that $F(q^o) < F(q^*)$ where q^o is a point with all $q_j^o \epsilon A_j$. Then convexity implies that for $0 < t < 1$,

$$F(t_q^o + (1-t)q^*) \leq tF(q^o) + (1-t)F(q^*) ,$$

which implies that

$$[F(q^*+t(q^o-q^*)) - F(q^*)]/t \leq F(q^o) - F(q^*) < 0 .$$

Since each function h_j is continuously differentiable, F is also.

+ D'Esopo, D., "A Convex Programming Procedure", <u>Naval Research Logistics Quarterly</u>, vol. 6, no. 1, March, 1969, pp. 33-42.

Let grad_j denote the gradient with respect to the two-dimensional vector q_j ; then the last inequality reads

$$\Sigma_j(\text{grad}_j F) \cdot (q_j^o - q_j^*) \leq F(q^o) - F(q^*) < 0$$

where the gradients are evaluated at a point $q^* + \theta_t t(q^o - q^*)$ with $0 \leq \theta_t \leq 1$. Since the gradients are continuous, we can let $t \longrightarrow (0^+)$ and conclude that the inequality also holds with the gradients evaluated at q^* . Thus for at least one j ,

$$(\text{grad}_j F)_{q*} \cdot (q_j^o - q_j^*) < 0 ,$$

so that F can be decreased by keeping all variables $\{q_J : J \neq j\}$ fixed and moving q_j from $q_j^* \epsilon A_j$ in the direction of $q_j^o \epsilon A_j$. But then q^* is not stable under the cyclic minimization process, and so could not arise as the limit of this process, contrary to hypothesis.

The preceding argument shows that the cyclic minimization method is theoretically adequate to solve our minimization problem. What makes it especially attractive here is that the sequence of two-dimensional minimization problems, to which it reduces the original problem, is of a special form facilitating solution. Let q_j be the distinguished variable at the t-th stage.

If $j = 1$, then the minimizing $\bar{q}_1 = q_1^{(t+1)}$ should clearly be chosen as a point on the boundary of polygon A_1 which is closest to $q_2^{(t)}$; similarly if $j = m$, then $\bar{q}_m = q_m^{(t+1)}$ should be chosen as a point on the boundary of polygon A_m which is closest to $q_{m-1}^{(t)}$. Suppose now that $1 < j < m$, so that $\bar{q}_j = q_j^{(t+1)}$ should be a point $q_j \epsilon A_j$ chosen to minimize

$$G^{(t)}(q_j) = h_{j-1}(|q_{j-1}^{(t)} - q_j|) + h_j(|q_{j+1}^{(t)} - q_j|) .$$

If the line segment $L^{(t)}$ joining $q_{j-1}^{(t)}$ and $q_{j+1}^{(t)}$ meets A_j, then $\bar{q}_j$ should clearly be chosen on the segment $L^{(t)} \cap A_j$, since any other choice in A_j could be replaced by its projection on $L^{(t)}$ with a reduction in the arguments of both h_{j-1} and h_j in $G^{(t)}(q_j)$. For the same reason, if $L^{(t)}$ does not meet A_j then $\bar{q}_j$ must be sought on those boundary segments of A_j which "face" $L^{(t)}$. Thus in any case, the two-dimensional minimization problem arising at the t-th stage reduces to one or more one-dimensional problems, with range either $L^{(t)} \cap A_j$ or a boundary segment of A_j.

The calculations described so far, it will be recalled, are to be performed for each permutation π of $\{1,2,\ldots,m\}$, or at least for all π corresponding to "reasonable" network configurations. We might refer to such a set of calculations (over all reasonable π) as a single <u>secondary computation</u>. Such a secondary computation, however, should in principle be carried out for each solution $\Sigma_o = \{S_j\}_1^m$ of the primary problem. Thus either there must not be too many such sets, or else the "principle" must be compromised and the process carried out only for a subset of the primary problem's solutions.

It should be noted at this point that the solution method for the primary problem, given in Section 2, may not be capable of yielding all minimum covers Σ_o. This is because of the reduction step (a), which eliminates a member $S_k \in \Sigma$ if it is a proper subset of some $S_\ell \in \Sigma$. While such a deletion is useful in searching for a single minimum cover, it certainly rules out discovering those minimum covers which include S_k. Such minimum covers may in fact exist, and if so it would really have been better for the secondary problem to have S_k rather than S_ℓ at hand, since this would leave one free to locate an access point anywhere in the larger set $\cap \{R_i : i \in S_k\}$ rather than the smaller $\cap \{R_i : i \in S_\ell\}$.

This complication may present a real difficulty. One could keep track of all applications of the reduction step, then go back after a minimum cover is found and consider the effects of "undoing" each of them, but this could prove quite complex unless the number of applications was small. One could omit use of the reduction step, but then the size of the minimum-cover problem might grow considerably. A third possibility begins with using the reduction step to arrive at a single minimum cover. Knowing the number m of members in this cover, one would then try without use of the reduction step to find all other covers with exactly m members.

3.4. APPENDIX. THE ONE-DIMENSIONAL PROBLEM

We consider here a simpler version of the problem discussed in 3.2, above, where the R_i are closed, finite intervals on the real line. It is again desired to find a point set $P = \{p(j)\}$ of minimum cardinality which meets every one of the R_i. The problem is easily solved in one dimension.

The first step in the solution of this problem is to eliminate all sets R_i such that there exists R_k, $k \neq i$, with $R_k \subset R_i$. For if $p(j) \in R_k$ then $p(j) \in R_i$, so that a solution to the reduced problem is a solution of the original. The elimination of all such R_i then leaves us with a <u>linear</u> set of R_i, in the sense that if $R_i = [a_i, b_i]$ and $R_k = [a_k, b_k]$, then either $a_i < a_k$ and $b_i < b_k$ or $a_i > a_k$ and $b_i > b_k$. Thus the R_i may be renumbered so that $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$, where $R_i = [a_i, b_i]$.

The second step in the solution is to apply a recursive algorithm for locating $p(j)$. The recursion step is now given.

ALGORITHM. Given an ordered set $\{R_i\}_1^n$ as described above, the following procedure locates the $p(j)$ optimally. Let $p(1) = b_1$. Eliminate from $\{R_i\}_1^n$ all R_i for which $p(1) \in R_i$. Repeat the procedure on the reduced set.

THEOREM. <u>The above algorithm produces a minimal-cardinality set</u> P.

PROOF. The proof is by induction on n, the number of R_i. Since the proof is obvious for $n = 1$, we prove it for general n, assuming its truth for values $1, 2, 3, \ldots, n-1$.

Given a set $\{R_i\}_1^n$ and a feasible set $P = \{p(j)\}$, there must be at least one $p(j) \epsilon R_1$. Pick one such point and, relabeling if necessary, call it $p(1)$. If P is optimal, then the set P' formed by replacing $p(1)$ with b_1 , if $p(1) \neq b_1$, is also optimal. For P' is feasible, since if $p(1) \epsilon R_i$ for any i then $b_1 \epsilon R_i$, by the linear ordering of the R_i . Also $|P| = |P'|$. Thus there exists an optimal P for which $p(1) = b_1$.

The proof then follows from the induction step by noting that $P - \{p(1)\}$ is a minimum cover for the set

$$\{R_i : p(1) \not\epsilon R_i\} ,$$

a set of cardinality less than n .

4. SUGGESTIONS FOR FUTURE WORK

The mathematical problems which arise in large geometrical positioning problems, such as those occurring in the papers presented in subsection 2.1 and section 3, have not been solved by any simple and/or quick mathematical methods. In general, one must look to heuristic, or partially heuristic methods in order to come up with near-optimal solutions to such problems. Therefore any research which facilitates the solution of such theoretical problems, especially when of a large size, would have applications to many practical situations.

Some, but not all, of the problems which come to mind for further effort in this respect are the following.

1. Traveling Salesman Problem. Given a set of n points $\{p_k\}$ with a matrix $D = \{d_{ij}\}$ of distances between point pairs, find an ordering of the p_k to minimize

$$\Sigma \{d_{ij} : p_i \text{ immediately follows } p_j \text{ in the ordering}\} .$$

(See [2] for more information on this problem.)

2. Given a set $\{p_j\}$ of points in the plane, find a finite set S of points, of a given cardinality, so that

$$\Sigma_j \alpha_j d(p_j, S) \qquad (\alpha_j > 0)$$

is minimized, where $d(p_j, S)$ is the minimum distance from p_j to S . (See for example [16].)

3. Given a matrix $A = \{a_{ij}\}$ of 0's and 1's, find a 0-1 integer solution vector X to the problem

$$\min |X|$$

subject to

$$AX \geq B ,$$

where $|X|$ is the sum of the components of X, and $B^T = (1,1,1,\ldots,1)$. (This is the problem of section 3.) (See [6].)

4. Same as 3, above, but A is now restricted merely to have non-negative entries, as is also B, and the objective is to

$$\min \quad CX$$

where C is also non-negative. (See [1].)

5. The problem of 2.4 for more general sets (than line segments).

6. More results on the network layout problem of 2.1, in the form stated.

7. A variation of 2.1 in which the Manhattan metric is used (i.e., distance between any two points (x,y) and (z,w) is given by $|x-z|+|y-w|$) .

8. The following variation of 2.1. Given a network N in the plane and a set of n points S , construct a new connected network N' containing N and S which minimizes the sum

$$\lambda \ell(N'-N) + \Sigma_{i,j} \lambda_{ij} \ell(P_{ij}) ,$$

where $\lambda > 0$, $\lambda_{ij} \geq 0$, ℓ is the length function, and P_{ij} is a shortest path between the i-th and j-th points of S in N' .

OR construct a new network N' containing S which minimizes the sum

$$\lambda \ell(N'-N) + \lambda'(N' \cap N) + \Sigma_{ij} \lambda_{ij} \ell(P_{ij}) ,$$

where $\lambda' > 0$, in addition to the above conditions.

9. "Build" a network of the type in 2.1, one stage at a time, possibly starting from a given network, so that the sum

$$\Sigma_t f(N_t) d_t$$

is minimized, where N_t is the network in its t-th stage, d_t is a discount factor, and f is some function of the network such as that given in 6,7, or 8, above. Constraints would be consistent with the particular problem.

10. Any of the above problems with more complex or general objective functions or constraints.

The above problems represent only a few of the mathematically difficult situations occurring in the study of accessibility. We may also mention some practical problems worthy of study.

1. A better determination of the connection between ready accessibility of the system to the user and the level of patronage.

2. Better data collection methods to use when the mathematical problems have been solved.

3. Better ways of allocating costs to projects and a more realistic determination of benefits (from a governmental point of view) than the somewhat intractable concept of "utility".

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